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Perturbative check on the Casimir energies of nondispersive dielectric spheres

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Abstract. For an optically dilute solid sphere of radius a and dielectric constant ε independent of frequency, the Casimir energy Δ is evaluated to second order in $\gamma \equiv (1 - 1/\varepsilon)$, subject to an exponential cut-off $1/\lambda$ on wavenumbers, using only standard perturbation theory and elementary mathematics. It is hoped that this can serve to elucidate other far more elaborate methods that aim to determine Δ exactly by summing zero-point energies. For the electromagnetic field, the perturbative result reads

$$\Delta(\mathrm{em}) = -\gamma \frac{3}{2\pi^2}\frac{V}{\lambda^4} + \gamma^2\left\{-\frac{3}{128\pi^2}\frac{V}{\lambda^4} + \frac{7}{360\pi^3}\frac{S}{\lambda^3} - \frac{1}{20\pi^2}\frac{1}{\lambda} + \frac{23}{1536\pi}\frac{1}{a}\right\} + \cdots$$

with V the volume and S the surface area. The term of order γ^2 is related in a simple way to the Casimir–Polder (retarded) potential between polarizable bodies. This relation also yields some insight into the net pressure on a thin spherical shell.

1. Introduction

Consider the self-energy Δ acquired, through its coupling to the quantized transverse Maxwell field, by a solid sphere (a ball) of radius a, having dielectric constant $\varepsilon = n^2$ independent of frequency. This tends to be approached through the total zero-point energy of the field in presence of the ball, requiring considerable mathematical sophistication. In contrast, we shall show that, for optically dilute media, ordinary perturbation theory suffices to determine Δ to second order as

$$\Delta = \gamma\Delta_1 + \gamma^2\Delta_2 + \cdots, \qquad 0 < \gamma \equiv (1 - 1/\varepsilon) \ll 1 \tag{1.1}$$

using wholly elementary mathematics, provided one adopts an explicit exponential cut-off $1/\lambda$ on wavenumbers. This expansion is proposed not as a step towards dealing with realistic materials, but purely as an aid towards charting other attempts on the same preliminary and somewhat artificial problem, which often discard divergences as it were behind the scene and sight unseen. We hope that the simplicity of the perturbative expressions might elicit some clarification/intercomparison of the various methods regularizing through analytic or dimensional (zeta-function) continuation: more advanced results aiming at exactitude can be checked by expanding them as in (1.1) and comparing coefficients.

It must be stressed that the problem without a cut-off is ill-defined, because Δ is divergent; and that no physical interpretation of cut-offs as simple as ours can be wholly compelling, because (a) real materials are dispersive, i.e. ε depends on the frequency ω, and tends to unity as ω rises well above any frequency characteristic of the material; and (b) dispersion is always due to some mechanical degrees of freedom (additional to those of the field), whose

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contributions to Δ cannot be mimicked reliably by starting with constant ε and then introducing cut-offs *a posteriori* into otherwise divergent integrals over frequency or wavenumber. Hence our programme as outlined above is modest not from virtue but from necessity: we study the constant-ε sphere with a cut-off in order to illuminate methods rather than to solve a real physical problem. Nevertheless, section 5 will hazard a possibly overoptimistic conjecture about the significance of the cut-off-independent parts of our results.

For simplicity, the body of this paper considers only a massless scalar field; the electromagnetic field is relegated to an appendix. Apart from purely numerical coefficients the results are the same for both. Section 2 recalls the basics of quantizing the field in the presence of a medium. Section 3 merely recalls the exact energy densities u in unbounded media, interesting on the widespread opinion (true for Δ_1 but false for Δ_2) that the component of Δ proportional to the volume $V = 4\pi a^3/3$ should be simply $V(u - u_{vac})$, where u_{vac} is the zero-point energy density *in vacuo*. Remarkably, both the sign and the magnitude of $(u - u_{vac})$ depend on whether one cuts off frequencies or wavenumbers.

Beyond this we consider only dilute media subject to (1.1). Sections 4.1 and 4.2 evaluate Δ_1 and Δ_2, respectively. One needs only plane waves: neither Bessel functions nor zeta functions ever appear. Section 5 makes some final comments: *conclusions* regarding such a largely technical exercise are best drawn by the reader.

Appendix A evaluates the one nontrivial integral we encounter, and appendix B adapts to electromagnetism. Appendix C links $\gamma^2\Delta_2$ to the familiar Casimir–Polder potential between molecules (quā-neutral polarizable objects). By exploiting this link appendix D then determines the net outward pressure on an infinitesimally thin spherical shell.

2. Quantizing the field

The Lagrangean density for our scalar field ϕ, the conjugate momentum Π, and the Hamiltonian density read†

$$\mathcal{L} = \tfrac{1}{2}\left\{\varepsilon(\boldsymbol{r})\dot{\phi}^2 - \left(\nabla\phi\right)^2\right\}, \qquad \Pi = \varepsilon(\boldsymbol{r})\dot{\phi}, \qquad \varepsilon(\boldsymbol{r}) \equiv \Theta(r-a) + \varepsilon\Theta(a-r)$$

$$\mathcal{H} = \tfrac{1}{2}\left\{\Pi^2/\varepsilon(\boldsymbol{r}) + \left(\nabla\phi\right)^2\right\} = \mathcal{H}_0 + \Delta\mathcal{H}, \qquad \mathcal{H}_0 = \tfrac{1}{2}\left\{\Pi^2 + \left(\nabla\phi\right)^2\right\}$$

where Θ is the Heaviside step function. Without the ball one would have $\varepsilon(\boldsymbol{r}) = 1$ everywhere. Accordingly, the interaction Hamiltonian is

$$\Delta H = -\tfrac{1}{2}\gamma \int_{r<a} \mathrm{d}^3 r\, \Pi^2(\boldsymbol{r}). \tag{2.1}$$

We shall need the normal-mode expansion for Π *in vacuo* and in the absence of the ball:

$$\Pi(\boldsymbol{r}) = -\mathrm{i}\int \mathrm{d}^3 k\, \frac{k^{1/2}}{4\pi^{3/2}} \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) a_{\boldsymbol{k}} + \text{H.c.} \tag{2.2}$$

where the $a_{\boldsymbol{k}}$ are the usual annihilation operators, such that $\left[a_{\boldsymbol{k}}, a^{+}_{\boldsymbol{k}'}\right] = \delta(\boldsymbol{k} - \boldsymbol{k}')$, and H.c. denotes Hermitean conjugate.

3. Zero-point energy densities in unbounded space

For plane waves in an unbounded medium one has

$$\omega = k/n, \qquad n \equiv \sqrt{\varepsilon};$$

† We use natural units, $\hbar = 1 = c$, and rationalized Gaussian units for the Maxwell field. We work in the Schrödinger picture.

choosing† an exponential cut-off the energy density reads

$$\begin{bmatrix} u_k \\ u_\omega \end{bmatrix} = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\omega}{2} \exp\left(-\begin{bmatrix} \lambda_k k \\ \lambda_\omega \omega \end{bmatrix} \right) \tag{3.1}$$

where the subscripts k (or ω) indicate cut-offs applied to wavenumber (or frequency). *In vacuo* there is no ambiguity, and we write u_{vac}, and λ without a subscript. We are interested only in cases where the appropriate λ is much smaller than any other pertinent length: here this requires only $\lambda \ll a$. Terms that vanish as $\lambda \to 0$ are dropped. This *no-cut-off limit* is always to be taken at the end of the calculation.

Evaluating (3.1) one finds

$$(u_{vac}, u_k, u_\omega) = (3/2\pi^2)\big(1/\lambda^4, 1/n\lambda_k^4, n^3/\lambda_\omega^4\big).$$

In particular, if one chooses a wavenumber cut-off and sets $\lambda_k = \lambda$, then

$$u_k - u_{vac} = -(1 - 1/n)\big(3/2\pi^2\lambda^4\big) = -\big(\tfrac{1}{2}\gamma + \tfrac{1}{8}\gamma^2 + \cdots\big)\big(3/2\pi^2\lambda^4\big) < 0 \tag{3.2}$$

and the radiative self-energy tightens the binding of the material (at least insofar as such cut-off-dependent terms echo any real physics). This choice is made for instance by Schwinger (1993). By contrast, if one chooses a frequency cut-off‡ and sets $\lambda_\omega = \lambda$, then

$$u_\omega - u_{vac} = (n^3 - 1)\big(3/2\pi^2\lambda^4\big) = \big(\tfrac{3}{2}\gamma + \tfrac{15}{8}\gamma^2 + \cdots\big)\big(3/2\pi^2\lambda^4\big) > 0 \tag{3.3}$$

and the radiative self-energy loosens the binding of the material.

The expansions in powers of γ are pertinent to optically dilute media that are our main concern: bear in mind for instance that

$$\varepsilon = n^2 = 1/(1-\gamma) \quad \Rightarrow \quad 1 - 1/n = 1 - \sqrt{1-\gamma}.$$

4. Perturbation theory

The unperturbed (zero-order) Hamiltonian is that for the free field in empty unbounded space, i.e. $H_0 = \int \mathrm{d}^3r\, \mathcal{H}_0$. Its eigenstates are the familiar plane-wave photon states.

The perturbation is ΔH, given by (2.1). To calculate its matrix elements between the zero-order states one needs only the expansion (2.2). We adopt a wavenumber cut-off $\exp(-\lambda k)$, which in such a perturbative approach seems the least implausible choice. Clearly, λ cannot well be smaller than the important absorption wavelengths of the material, which otherwise our nondispersive model ignores altogether. (Equally obvious though usually less restrictive is the condition that λ must be well above the intermolecular or lattice spacing, since waves shorter than this do not see the medium as continuous.)

† The exponential cut-off is adopted for convenience: other choices presumably lead to similar conclusions, subject of course to the caution in section 1 regarding dispersion. (However, it would be disingenuous to try and hide that explicit calculations are incomparably easier with the exponential than with any other cut-off.) Alternatives include the condition $k < 1/\lambda$; this, like ours, is effectively a cut-off imposed on Fock space. More plausibly perhaps one might require the refractive index $n = k/\omega$ to reduce to unity at high k or high ω. Such cut-offs via n are discussed instructively by Carlson *et al* (1997).

‡ This option when imposed through a refractive index is criticized by Carlson *et al* (1997), though it is not immediately clear whether their strictures apply to our more primitive variant.

4.1. First-order energy shift

The shift $\gamma\Delta_1$ is simply the expectation value of the perturbation in the zero-order state, i.e. the unperturbed-vacuum expectation value of ΔH:

$$\gamma\Delta_1 = \langle 0|\Delta H|0\rangle = -\tfrac{1}{2}\gamma\int_{r<a} \mathrm{d}^3 r\,\langle 0|\Pi^2|0\rangle$$
$$= -\tfrac{1}{2}\gamma\int \mathrm{d}^3 k\,\frac{k}{16\pi^3}\exp(-\lambda k)V = -\frac{3\gamma}{4\pi^2\lambda^4}V \tag{4.1}$$

where the third step relies on the fact that the zero-order expression $\langle 0|\Pi^2|0\rangle$ is independent of position. Notice that $\gamma\Delta_1$ is just V multiplied by $u_k - u_{vac}$ from (3.2) taken to first order in γ. Being proportional to volume for all shapes, $\gamma\Delta_1$ is irrelevant to forces tending merely to shift or even to distort bodies without expanding them.

4.2. Second-order energy shift

Since ΔH links the vacuum only to two-photon states $|\boldsymbol{k},\boldsymbol{k}'\rangle$, the second-order shift is

$$\gamma^2\Delta_2 = -\tfrac{1}{2}\int\!\!\int \mathrm{d}^3 k\,\mathrm{d}^3 k'\,\frac{|\langle \boldsymbol{k},\boldsymbol{k}'|\Delta H|0\rangle|^2}{k+k'}\exp\big[-\lambda(k+k')\big].$$

The prefactor $\frac{1}{2}$ compensates for double-counting the states $|\boldsymbol{k},\boldsymbol{k}'\rangle = |\boldsymbol{k}',\boldsymbol{k}\rangle$.

Given our exponential cut-off, Δ_2 can be evaluated quite easily. First one writes $1/(k+k') = \int_0^\infty \mathrm{d}\mu\,\exp\big[-\mu(k+k')\big]$ and sets $\xi \equiv (\mu+\lambda)$, which leads to

$$\frac{\exp\big[-\lambda(k+k')\big]}{k+k'} = \int_\lambda^\infty \mathrm{d}\xi\,\exp\big[-\xi(k+k')\big].$$

Second, one uses (2.2) to write out $\langle \boldsymbol{k},\boldsymbol{k}'|\Delta H|0\rangle$ and its complex conjugate as integrals $\int \mathrm{d}^3 r\,\ldots$ and $\int \mathrm{d}^3 r'\ldots$, respectively. This in effect factors the integrand, and allows us to reverse the order of the integrations over positions and wavevectors:

$$\Delta_2 = -\frac{1}{2(16\pi^3)^2}\int_{r<a}\int_{r'<a} \mathrm{d}^3 r\,\mathrm{d}^3 r'\int_\lambda^\infty \mathrm{d}\xi\left\{\int \mathrm{d}^3 k\,k\,\exp\big[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}') - \xi k\big]\right\}^2. \tag{4.2}$$

Note that $\int \mathrm{d}^3 k\ldots$ is real; and that Bose–Einstein factors of 2 from each matrix element have cancelled the factors $\frac{1}{2}$ of ΔH.

Third, one introduces the coordinates

$$\boldsymbol{R} \equiv \boldsymbol{r}+\boldsymbol{r}' \qquad \boldsymbol{\rho} \equiv \boldsymbol{r}-\boldsymbol{r}' \qquad |\partial(\boldsymbol{r},\boldsymbol{r}')/\partial(\boldsymbol{R},\boldsymbol{\rho})| = \tfrac{1}{8} \tag{4.3}$$

and the scaled auxiliary variable

$$x \equiv \xi/\rho. \tag{4.4}$$

Then

$$\int \mathrm{d}^3 k\,k\,\exp\big[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\rho} - \xi k\big] = \frac{8\pi}{\rho^4}\frac{\big(3x^2-1\big)}{(x^2+1)^3} \tag{4.5}$$

leads to

$$\Delta_2 = -\tfrac{1}{2}\int_{r<a}\int_{r'<a}\frac{\mathrm{d}^3 r\,\mathrm{d}^3 r'}{\rho^7}\frac{1}{4\pi^4}\int_{\lambda/\rho}^\infty \mathrm{d}x\,G(x), \qquad G(x) \equiv \frac{\big(3x^2-1\big)^2}{(x^2+1)^6}. \tag{4.6}$$

(The obvious interpretation of $\gamma^2\Delta_2$ in terms of a local potential is spelled out in appendix C.)

Fourth, one changes the integration variables to $\boldsymbol{\rho}$, $\boldsymbol{R}$, performs $\int \mathrm{d}^3R\ldots$, and rescales from ρ to

$$\sigma \equiv \rho/2a.$$

It is only here that not every step may be obvious, and the calculation is done in appendix A. The crucial result† is

$$\int_{r<a}\int_{r'<a} \mathrm{d}^3r\,\mathrm{d}^3r'\,f(\rho) = \tfrac{1}{3}2^7\pi^2a^6\int_0^1 \mathrm{d}\sigma\,\sigma^2\big\{1-\tfrac{1}{2}3\sigma+\tfrac{1}{2}\sigma^3\big\}f(\rho=2a\sigma) \tag{4.7}$$

with f any function of $\rho = |\boldsymbol{\rho}|$ alone. Applied to (4.6) this yields

$$\Delta_2 = -\frac{1}{24\pi^2a}\int_0^1 \mathrm{d}\sigma\left\{\frac{1}{\sigma^5}-\frac{3}{2\sigma^4}+\frac{1}{2\sigma^2}\right\}\int_{\lambda/2a\sigma}^{\infty}\mathrm{d}x\,G(x).$$

Finally, on inspecting the integration region in the (σ, x) plane, one can reverse the order of integrations and perform $\int \mathrm{d}\sigma\ldots$ first:

$$\begin{aligned}\Delta_2 &= -\frac{1}{24\pi^2a}\int_{\lambda/2a}^{\infty}\mathrm{d}x\,G(x)\int_{\lambda/2ax}^{1}\mathrm{d}\sigma\left\{\frac{1}{\sigma^5}-\frac{3}{2\sigma^4}+\frac{1}{2\sigma^2}\right\}\\ &= \frac{1}{24\pi^2a}\int_{\lambda/2a}^{\infty}\mathrm{d}x\,G(x)\left\{-4\left(\frac{ax}{\lambda}\right)^4+4\left(\frac{ax}{\lambda}\right)^3-\left(\frac{ax}{\lambda}\right)+\frac{1}{4}\right\}.\end{aligned} \tag{4.8}$$

This expression is still exact. But as $\lambda \to 0$ (in the sense, explained earlier, that $\lambda/a \ll 1$), we can extend the integration down to $x = 0$, since the difference vanishes with λ. To see this, note that $G(0) = 1$, whence

$$\int_0^{\lambda/2a}\mathrm{d}x\,G(x)\left(\frac{ax}{\lambda}\right)^4 \sim \left(\frac{a}{\lambda}\right)^4\frac{1}{5}\left(\frac{\lambda}{2a}\right)^5 \sim \frac{\lambda}{a} \to 0$$

and similarly for the other terms within the braces in (4.8). Accordingly

$$\Delta_2 = \frac{1}{24\pi^2a}\int_0^{\infty}\mathrm{d}x\,G^2(x)\left\{-4\left(\frac{ax}{\lambda}\right)^4+4\left(\frac{ax}{\lambda}\right)^3-\left(\frac{ax}{\lambda}\right)+\frac{1}{4}\right\} \tag{4.9}$$

which integrates to

$$\begin{aligned}\Delta_2 &= \frac{1}{24\pi^2a}\left\{-4\left(\frac{a}{\lambda}\right)^4\frac{3\pi}{32}+4\left(\frac{a}{\lambda}\right)^3\frac{3}{20}-\left(\frac{a}{\lambda}\right)\frac{1}{10}+\frac{1}{4}\frac{3\pi}{32}\right\},\\ \gamma^2\Delta_2 &\equiv \gamma^2\Delta_2(\text{scalar}) = \gamma^2\left\{-\frac{1}{64\pi}\frac{a^3}{\lambda^4}+\frac{1}{40\pi^2}\frac{a^2}{\lambda^3}-\frac{1}{240\pi^2}\frac{1}{\lambda}+\frac{1}{1024\pi}\frac{1}{a}\right\}.\end{aligned} \tag{4.10}$$

In terms of volume $V = 4\pi a^3/3$, surface $S = 4\pi a^2$, and radius a this reads

$$\gamma^2\Delta_2(\text{scalar}) = \gamma^2\left\{-\frac{3}{256\pi^2}\frac{V}{\lambda^4}+\frac{1}{160\pi^3}\frac{S}{\lambda^3}-\frac{1}{240\pi^2}\frac{1}{\lambda}+\frac{1}{1024\pi}\frac{1}{a}\right\}. \tag{4.11}$$

The analogous expression for the Maxwell field is found in appendix B:

$$\gamma^2\Delta_2(\text{em}) = \gamma^2\left\{-\frac{3}{128\pi^2}\frac{V}{\lambda^4}+\frac{7}{360\pi^3}\frac{S}{\lambda^3}-\frac{1}{20\pi^2}\frac{1}{\lambda}+\frac{23}{1536\pi}\frac{1}{a}\right\}. \tag{4.12}$$

† It is reassuring to verify that $f = 1$ gives $J = (4\pi a^3/3)^2$, and that $f = 1/2\rho = 1/4a\sigma$ gives $J = (4\pi a^3/3)^2(3/5a)$, the correct Coulomb energy for a uniformly charged sphere with unit charge density.

5. Comments

- The leading (most divergent) terms of the $\gamma^2\Delta_2$ are proportional to V. They combine naturally with the first-order shifts; thus

$$\Delta(\text{scalar}) = -\frac{V}{\lambda^4}\left\{\frac{3\gamma}{4\pi^2} + \frac{3\gamma^2}{256\pi^2} + \cdots\right\} + (\text{less divergent terms}).$$

  This may be compared with the expression $V(u_k - u_{vac})$ constructed from the energy densities of unbounded regions, whose expansion is given by (3.2) as

$$V(u_k - u_{vac}) = -\frac{V}{\lambda^4}\left(\frac{3\gamma}{4\pi^2} + \frac{3\gamma^2}{16\pi^2} + \cdots\right).$$

  Thus $V(u_k - u_{vac})$ coincides with the V/λ^4-proportional part of the true shift $\Delta(\text{scalar})$ to order γ but not to order γ^2. Although the difference may surprise at first sight, no theorem says that it must vanish.
- There is a positive surface energy.
- For many purposes the components of Δ proportional to V and to S would be combined with other contributions to the bulk and to the surface energies of the material, and play no further role if one uses the measured values. The last bullet below reverts to this point.
- There is no component proportional to $\gamma^2 a/\lambda^2$, such as might have arisen from multiplying S with the curvature.
- The third term in (4.11), (4.12) is independent of the radius, while the fourth appears to diverge as $a \to 0$. This seems paradoxical, because if there is no sphere ($a = 0$), then there ought to be no energy shift. But there is no real contradiction: to derive the Δ_2 we have assumed $\lambda \ll a$, and this assumption fails as $a \to 0$.
- The numerical coefficients of the terms featuring λ evidently depend on the kind of cut-off one has chosen. Therefore, as between results from different regularization methods, all that it makes sense to compare is the presence or absence of components with given powers of $1/\lambda$; the signs of these divergent components; and the finite term independent of λ, call it Δ^*. For calculations that sidestep divergences without ever identifying them, only Δ^* can serve as a check. But *discrepancies regarding Δ^* inevitably diagnose either errors in the calculation or a basic lack of mathematical definition in the problem*. Our last bullet elaborates this too.
- As a second-order perturbation of the ground-state energy, $\gamma^2\Delta_2$ must be negative, as it is in virtue of its dominant V-proportional component. But its finite component is positive:

$$\gamma^2\Delta_2^*(\text{scalar}) = \gamma^2/1024\pi a, \qquad \gamma^2\Delta_2^*(\text{em}) = 23\gamma^2/1536\pi a. \tag{5.1}$$

  There are no such finite terms of order γ.
- *Summations of zero-point energies* have produced various expressions.

  (a) Milton (1980), equation (44), gave $\Delta^*(\text{em}) = -\gamma^2/256a$, differing from (5.1) in sign and magnitude. (See also Milton (1996) equation (51); Milton and Ng (1997) equation (7.6).)

  (b) Brevik *et al* (1998) in their equation (3.5) gave $\Delta^*(\text{em}) = (\gamma^2/a)3/1024$, up to higher orders in γ. Compare their coefficient $3/1024 = 0.002\,93$ with our $23/1536\pi = 0.004\,766$: their approximations have produced the right sign and order of magnitude, but not quite the right number.

  (c) As regards the divergent components, Milton and Ng (1997) appear to ignore volume-proportional terms, but their equation (7.10) gives a surface term as $-\gamma^2a^2/256\lambda^3$, where we have substituted our $1/\lambda$ for their frequency cut-off ω_0. Here too sign and magnitude differ from the second term of our result (4.12), which yields $7\gamma^2 a^2/90\pi^2\lambda^3$.

(d) In a recent preprint, Brevik and Marachevsky (1998) correct some earlier oversights, and estimate† $\Delta^*(\mathrm{em}) = \left(\gamma^2/a\right)C$, with $0.004\,03 < C < 0.004\,85$, compatibly with $C = 0.004\,766$ from (5.1).

(e) Brevik *et al* (1998), correcting (a), give $\Delta^*(\mathrm{em}) = \left(\gamma^2/a\right)C$ with $C = 0.004\,767$ calculated numerically, an excellent approximation to (5.1).

• As regards divergence or convergence with vanishing λ, one might perhaps formulate an optimistic yet not totally absurd conjecture as follows. It should be read in the light of the caution already voiced in section 1.

(i) Since divergences stem from high wavenumbers, they pertain, loosely speaking, to physics at small distances: for instance, equation (C.2) in appendix C interprets $\gamma^2\Delta_2$ in terms of a pairwise potential between polarizable volume elements separated by ρ, which only the finite value of λ prevents from diverging nonintegrably as‡ $\rho \to 0$. But the true potential for $\rho \preceq \lambda$ is governed by the direct electrostatic couplings between molecules and/or charge carriers. Better models would supply explicit Hamiltonians for such short-distance physics, and would also describe the coupling to the quantized field more realistically than do our ΔH.

(ii) Short of such input, our kind of problem necessarily runs into a dilemma allied to a paradox. By tradition, 'Casimir effects' denote *macroscopic* forces and energy shifts; yet for connected bodies the macroscopic must be matched to *microscopic* physics, and no purely macroscopic model can be guaranteed in advance to reproduce the results of this matching adequately for whatever purpose is in hand. One faces questions reminiscent of those that for atoms are answered pretty informatively by the so-called nonrelativistic (Bethe) theory of the Lamb shift. Answers to these questions would tell us, for instance, whether *all* divergent terms can be absorbed by renormalizing material properties, or whether this is possible only for those proportional to V or to S.

(iii) To escape from this dilemma, one can try introducing some vestigial microscopic physics, say through model Hamiltonians for materials with a reasonably dispersive dielectric response. Only calculation reveals whether in any particular model the parameters specifying the dispersion allow one to dispense with cut-offs altogether. However, it seems to be widely if tacitly expected that in a reasonably wide class of models they do; and also that one will then be able to identify certain dispersion-independent terms common to all such models. If so, then the finite contributions Δ^* found in this paper determine these common terms to order γ^2.

Appendix A. The double integral over the ball

Equation (4.6) requires an integral of the type

$$J \equiv \int_{r<a}\int_{r'<a} \mathrm{d}^3r\,\mathrm{d}^3r'\,f(\rho) = \tfrac{1}{8}\int \mathrm{d}^3\rho\,f(\rho)\int \mathrm{d}^3R$$

with $\boldsymbol{\rho}$, $\boldsymbol{R}$ defined by (4.3), whence $\boldsymbol{r} = (\boldsymbol{R}+\boldsymbol{\rho})/2$, $\boldsymbol{r}' = (\boldsymbol{R}-\boldsymbol{\rho})/2$. The problem is to determine the ranges of the new integration variables. Let θ be the angle between $\boldsymbol{R}$ and $\boldsymbol{\rho}$;

† Their equation (30) estimates the nondispersive surface force F, which they link to the energy through $\gamma\Delta^* = 4\pi a^3 F$ (private communication from Professor Brevik). Recall that to leading order $(n-1) = \gamma/2$.

‡ Cut-offs are required only for connected bodies: the mutual Casimir energy of two disconnected bodies needs none.

then

$$\left(r^2, r'^2\right) = \left(R^2 + \rho^2 \pm 2R\rho\cos\theta\right)/4 \leqslant a^2.$$

Since both conditions apply, we impose the more stringent, replacing $\pm\cos\theta$ by $|\cos\theta| \equiv \mu$, with $0 \leqslant \mu \leqslant 1$. In terms of the scaled coordinates $\boldsymbol{S} = \boldsymbol{R}/2a$, $\boldsymbol{\sigma} = \boldsymbol{\rho}/2a$ (whence $0 \leqslant S, \sigma \leqslant 1$) this leads to

$$S^2 + 2\sigma\mu S - (1 - \sigma^2) < 0$$

which holds between $S = 0$ and the root

$$S_1 = -\sigma\mu + \sqrt{1 - \sigma^2 + \sigma^2\mu^2}.$$

Accordingly

$$\begin{aligned} J &= \tfrac{1}{8}(2a)^6 4\pi \int_0^1 \mathrm{d}\sigma\, \sigma^2 f\, 4\pi \int_0^1 \mathrm{d}\mu \int_0^{S_1} \mathrm{d}S\, S^2 \\ &= \frac{(2a)^6(4\pi)^2}{8\times 3} \int_0^1 \mathrm{d}\sigma\, \sigma^2 f \int_0^1 \mathrm{d}\mu \left\{\sqrt{1-\sigma^2+\sigma^2\mu^2} - \sigma\mu\right\}^3. \end{aligned} \qquad \text{(A.1)}$$

On evaluating $\int \mathrm{d}\mu \ldots$ this yields (4.7).

Appendix B. The Maxwell field

In vacuo the plane-wave expansion of the transverse electric displacement in the Coulomb gauge ($\nabla \cdot \boldsymbol{A} = 0$, $\boldsymbol{E} = \boldsymbol{D} = -\dot{\boldsymbol{A}}$) reads

$$\boldsymbol{D} = \mathrm{i} \sum_{s=1,2} \int \mathrm{d}^3k\, \frac{k^{1/2}}{4\pi^{3/2}} \boldsymbol{\epsilon}_{\boldsymbol{k}s} \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) a_{\boldsymbol{k}s} + \mathrm{H.c.}$$

and the interaction† with the ball is

$$\Delta H = -\tfrac{1}{2}\gamma \int_{r<a} \mathrm{d}^3r\, \boldsymbol{D}^2(\boldsymbol{r}). \qquad \text{(B.1)}$$

The polarization vectors are taken as real, with

$$\boldsymbol{\epsilon}_{\boldsymbol{k}s}\boldsymbol{\epsilon}_{\boldsymbol{k}s'} = \delta_{ss'}, \qquad \boldsymbol{k}\boldsymbol{\epsilon}_{\boldsymbol{k}s} = 0, \qquad \sum_s \epsilon_{\boldsymbol{k}si}\epsilon_{\boldsymbol{k}sj} = \delta_{ij} - k_i k_j/k^2.$$

To first order in γ each polarization ($s = 1, 2$) contributes as for a scalar field, so that $\Delta_1(\mathrm{em}) = 2\Delta_1(\mathrm{scalar})$.

To second order in γ, one finds for $\Delta_2(\mathrm{em})$ an expression obtainable from (4.2) by replacing

$$K(\mathrm{scalar}) \equiv \left\{\int \mathrm{d}^3k\, k \exp[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\rho} - \xi k]\right\}^2 \to$$

$$K(\mathrm{em}) \equiv \iint \mathrm{d}^3k\, \mathrm{d}^3k'\, kk' \exp[-\xi(k+k')] \exp[\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{\rho}] \sum_s \sum_{s'} (\epsilon_{ks}.\epsilon_{k's'})^2.$$

Now, using the dummy suffix convention,

$$\sum_s \sum_{s'} (\epsilon_{ks}.\epsilon_{k's'})^2 = \left(\delta_{ij} - k_i k_j/k^2\right)\left(\delta_{ij} - k'_i k'_j/k'^2\right) = 1 + \left(k_i k_j/k^2\right)\left(k'_i k'_j/k'^2\right) \qquad \text{(B.2)}$$

† In the Coulomb gauge the longitudinal components of the field are not quantized: they stem from the fluctuations of the atomic dipoles, and are responsible for the Van der Waals contributions to the self-energy.

making it convenient to define the symmetric tensor

$$M_{ij} \equiv \int \mathrm{d}^3k\, k\frac{k_i k_j}{k^2}\exp[-\xi k + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\rho}] = 8\pi\left\{\frac{\delta_{ij}(\xi^2+\rho^2)-\rho_i\rho_j}{(\xi^2+\rho^2)^3}\right\}. \tag{B.3}$$

Thus

$$K(\mathrm{em}) = M_{ii}^2 + M_{ij}M_{ij}. \tag{B.4}$$

Evidently

$$K(\mathrm{scalar}) = M_{ii}^2 = \frac{(8\pi)^2\big(3\xi^2-\rho^2\big)^2}{(\xi^2+\rho^2)^6} = \frac{(8\pi)^2}{\rho^8}\frac{\big(3x^2-1\big)^2}{(x^2+1)^6} = \frac{(8\pi)^2}{\rho^8}G(x) \tag{B.5}$$

with $x = \xi/\rho$ as in (4.4). The second term of (B.4) is

$$M_{ij}M_{ij} = \frac{(8\pi)^2}{\rho^8}\frac{(3x^4-2x^2+11)}{(x^2+1)^6}. \tag{B.6}$$

Combining the two terms in (B.4) we find

$$K(\mathrm{em}) = \frac{(8\pi)^2}{\rho^8}G_{\mathrm{em}}(x), \qquad G_{\mathrm{em}}(x) \equiv \frac{12x^4-8x^2+12}{(x^2+1)^6}. \tag{B.7}$$

Thus the electromagnetic are obtained from the scalar expressions simply by replacing $G(x) = (3x^2-1)^2/(x^2+1)^6$ with $G_{\mathrm{em}}(x)$. Doing so in (4.9) yields

$$\Delta_2(\mathrm{em}) = \frac{1}{24\pi^2 a}\int_0^\infty \mathrm{d}x\, G_{\mathrm{em}}(x)\left\{-4\left(\frac{ax}{\lambda}\right)^4 + 4\left(\frac{ax}{\lambda}\right)^3 - \left(\frac{ax}{\lambda}\right) + \frac{1}{4}\right\};$$

this integrates to

$$\Delta_2(\mathrm{em}) = \frac{1}{24\pi^2 a}\left\{-4\left(\frac{a}{\lambda}\right)^4\frac{3\pi}{16} + 4\left(\frac{a}{\lambda}\right)^3\frac{7}{15} - \left(\frac{a}{\lambda}\right)\frac{6}{5} + \frac{1}{4}\cdot\frac{23\pi}{16}\right\}$$

which leads to (4.12).

Appendix C. The Casimir–Polder connection

In hindsight, equation (4.6) adapted to the Maxwell field may be rearranged to yield

$$\gamma^2\Delta_2 = -\tfrac{1}{2}\int_{r<a}\int_{r'<a}\left(\frac{\gamma\,\mathrm{d}^3r}{4\pi}\right)\left(\frac{\gamma\,\mathrm{d}^3r'}{4\pi}\right)\frac{1}{\rho^7}\frac{4}{\pi^2}\int_{\lambda/\rho}^\infty \mathrm{d}x\, G_{\mathrm{em}}(x). \tag{C.1}$$

But $\gamma\,\mathrm{d}^3r/4\pi \equiv \mathrm{d}\alpha$ is just the (static) polarizability† of the volume element d^3r containing our dilute dielectric with $\varepsilon = 1 + \gamma + \cdots$. Thus (C.1) may be interpreted as asserting that between two objects with polarizabilities $\mathrm{d}\alpha$ and $\mathrm{d}\alpha'$ a distance ρ apart, there acts an attractive potential

$$\mathrm{d}\alpha\,\mathrm{d}\alpha'\,U(\rho), \qquad U(\rho) \equiv -\frac{1}{\rho^7}\frac{4}{\pi^2}\int_{\lambda/\rho}^\infty \mathrm{d}x\, G_{\mathrm{em}}(x). \tag{C.2}$$

† We define polarizability so that in volume $\mathrm{d}V$ an electric field $\boldsymbol{E}$ induces a dipole moment $\mathrm{d}\boldsymbol{P} = \boldsymbol{E}\,\mathrm{d}V(\varepsilon-1)/4\pi = \boldsymbol{E}\,\mathrm{d}\alpha$. The old-fashioned 4π here conforms to the definition of molecular polarizabilities α featured in the standard expression for the Casimir–Polder potential.

As $\rho/\lambda \to \infty$ the lower limit may be replaced by zero, and

$$\frac{4}{\pi^2}\int_0^\infty \mathrm{d}x\, G_{\mathrm{em}}(x) = \frac{4}{\pi^2}\cdot\frac{23\pi}{16} = \frac{23}{4\pi} \tag{C.3}$$

reduces (C.2) to the familiar Casimir–Polder (retarded) potential $-23\mathrm{d}\alpha\,\mathrm{d}\alpha'/4\pi\rho^7$. By contrast, $U(\rho/\lambda \to 0) = -48/7\pi^2\lambda^7$. The preamble to section 4 explained that λ should be of the order of the typical absorption wavelengths, and much larger than the intermolecular spacing. For $\rho \ll \lambda$ the retarded potential (C.2) is therefore swamped by the (electrostatic) Van der Waals interactions†, of order $-\mathrm{d}\alpha\,\mathrm{d}\alpha'/4\pi\lambda\rho^6$, whose contributions to the total binding far outweigh $\gamma^2\Delta_2$.

The energy shift calculable directly from two-body Casimir–Polder potentials (rather than by summing zero-point energies) has already been considered by Milton and Ng (1998). Using dimensional continuation, and disregarding divergent parts, they extract a finite shift, their equation (3.17), which coincides exactly with $\gamma\Delta_2^*(\mathrm{em})$ in (5.1).

Appendix D. The net pressure on a thin spherical shell

Equations (C.1) and (C.2) can be applied to a thin spherical shell of radius a and thickness δa. For simplicity we idealize by taking $\delta a \ll \lambda \ll a$, as if the electromagnetic interaction-Hamiltonian density were $-\Gamma\delta(r-a)\boldsymbol{D}^2/2$, with $\Gamma \equiv \gamma\delta a$ treated as a single input parameter in its own right. Then to order Γ^2 the net outward pressure (force per unit area), call it P, can be determined directly from the pairwise intervolume forces, evidently proportional to $F(\rho) \equiv -\partial U/\partial\rho$. (The first-order shift $\gamma\Delta_1$ is manifestly irrelevant, depending as it does only on the total volume of the material. Several other distractions too are sidestepped by calculating P directly rather than through the energy.)

We omit the details; in the integration over the surface one changes the variable from polar angle to ρ, and finds

$$\begin{aligned} P(\mathrm{em}) &= \left(\frac{\Gamma}{4\pi}\right)^2 \frac{\pi}{a}\int_0^{2a}\mathrm{d}\rho\,\rho^2 F(\rho) = -\left(\frac{\Gamma}{4\pi}\right)^2\frac{\pi}{a}\left\{(2a)^2U(2a) - 2\int_0^{2a}\mathrm{d}\rho\,\rho U(\rho)\right\}\\ &= \left(\frac{\Gamma}{4\pi}\right)^2\frac{4}{\pi a}\left(\frac{1}{2a}\right)^5\int_0^\infty \mathrm{d}x\,G_{\mathrm{em}}(x)\frac{1}{5}\left\{-2\left(\frac{2ax}{\lambda}\right)^5 + 7\right\}\\ &\quad + (\text{terms vanishing with }\lambda), \end{aligned}$$

$$P(\mathrm{em}) = \left(\frac{\Gamma}{4\pi}\right)^2\frac{1}{a^6}\left\{-\frac{48}{25\pi}\left(\frac{a}{\lambda}\right)^5 + \frac{161}{640}\right\}. \tag{D.1}$$

Concentrating all the material into an infinitesimally thin shell has aggravated the divergence to $(a/\lambda)^5$; finite thickness would produce nothing worse than the $(a/\lambda)^4$ encountered for solid spheres.

Since the forces are attractive, P must be negative, as it is in virtue of its dominant (divergent) component: the shell tends to collapse. On the other hand, the finite term is positive. If one accepts the optimistic conjecture at the end of section 5, whereby a proper calculation would absorb *all* divergent components through renormalizations of the various material properties of the medium, then from the Casimir scene the first (the attractive) term in (D.1) might vanish; and the second (the repulsive) term might turn out to be the correct adaptation, to our feebly polarizable shell, of Boyer's classic result for a perfectly reflecting

† Formally, our Hamiltonian (B.1) admits these dominant contributions through the unquantized longitudinal components of $\boldsymbol{D}$.

shell, namely that it too experiences a net outward pressure on account purely of the quantized Maxwell field. (For references and a very clear discussion see Bowers and Hagen (1998).) In that case there would exist a well defined interpolating function $P(\mathrm{em}, a, \Gamma)$, such that

$$P(\mathrm{em}, a, \Gamma \to 0) = \frac{161}{640}\left(\frac{\Gamma}{4\pi}\right)^2 \frac{1}{a^6}, \qquad P(\mathrm{em}, a, \Gamma \to \infty) = \frac{0.092\,35}{8\pi a^4} \qquad (?)$$

where the rightmost expression quotes from Boyer (1968).

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